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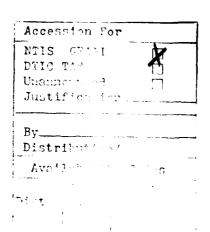Management Science
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ON STOCHASTIC COMPARISON OF RANDOM VECTORS

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ABSTRACT

We provide sufficient conditions under which two random vectors could be stochastically compared using the standard construction. These conditons are weaker than those discussed by Arjas and Lehtonen (1978) and Veinott (1965). Using these conditions we present extensions of (i) a result of Block, Bueno, Savits and Shaked (1984) concerning the stochastic monotonicity of independent and identically distributed random variables conditioned on their partial order statistics, and (ii) a theorem of Efron (1965) regarding an increasing property of Pólya frequency functions. Applications of these extensions are also pointed out.

Key words: Random vectors, stochastic ordering, negative dependence, Pólya frequency function, component cannibalization.

1. INTRODUCTION

Let X and Y be two real valued random variables with survival functions $\bar{F}$ and $\bar{G}$ respectively. Then X is said to be stochastically larger than Y if:

$$(1.1) \qquad \bar{F}(t) \geq \bar{G}(t) \quad , \quad t \in R$$

and is written $X \geq_{st} Y$ [When equality holds in (1.1) for all values of t we write $X \overset{st}{=} Y$. That is, they are equal in law]. Once $\bar{F}$ and $\bar{G}$ are given it is usually easy to verify (1.1). The natural extension of (1.1) for finite or infinite dimensional random vectors is as follows [e.g. Kamae, Krengel and O'Brien (1977)]: Let $\underline{X} = (X_1, X_2, \ldots)$ and $\underline{Y} = (Y_1, Y_2, \ldots)$ be two n-component random vectors $[n \geq 1$ or $n = \infty]$ with survival functions $\bar{F}$ and $\bar{G}$ respectively. Then $\underline{X}$ is said to be stochastically larger than $\underline{Y}$ if

$$(1.2) \qquad P\{\underline{X} \in A\} \geq P\{\underline{Y} \in A\} \text{ for every increasing set } A \in R^n \ ,$$

and is written $\underline{X} \geq_{st} \underline{Y}$ [When equality holds in (1.2) for all increasing sets $A \in R^n$ we write $\underline{X} \overset{st}{=} \underline{Y}$]. In this paper 'increasing' stands for 'nondecreasing' and 'decreasing' for 'nonincreasing'. Unfortunately, even with $\bar{F}$ and $\bar{G}$ explicitly specified it is usually not very easy to verify (1.2). However, in light of

Lemma 1.1: $\underline{X} \geq_{st} \underline{Y}$ if and only if there exist two random vectors $\hat{\underline{X}}$ and $\hat{\underline{Y}}$ defined on a common probability space such that $P\{\hat{\underline{X}} \geq \hat{\underline{Y}}\} = 1$ and $\underline{X} \overset{st}{=} \hat{\underline{X}}$ and $\underline{Y} \overset{st}{=} \hat{\underline{Y}}$,

[e.g. Kamae, Krengel and O'Brien (1977)], effort has been made to stochastically compare random vectors by constructing them on a common probability space. In this respect three alternative constructions have been used. They are (i) <u>standard construction</u> [e.g. Arjas and Lehtonen (1978)], (ii) <u>non-homogeneous Poisson process (NHPP) construction</u> [e.g. Shaked and Shanthikumar (1984)], and (iii) <u>total hazard construction</u> [e.g. Norros (1984), Shaked and Shanthikumar (1985)]. Since we will be using the standard construction we will describe it here. Let

$$\bar{F}_1(t) = P\{X_1 > t\} \quad , \quad t \, \varepsilon \, R$$

and

$$\bar{F}_j(t|\underline{x}_{j-1}) = P\{X_j > t | X_1 = x_1, \ X_2 = x_2, \ldots, X_{j-1} = x_{j-1}\},$$

$$x_1, x_2, \ldots, x_{j-1}, \ t \, \varepsilon \, R; \quad j \geqq 2 \ .$$

Note that the dimension of the vector $\underline{x}_{j-1}$ will vary depending on where it is used. In $\bar{F}_j(\cdot|\underline{x}_{j-1})$, $\underline{x}_{j-1}$ will represent $(x_1, x_2, \ldots, x_{j-1})$. We will follow this convention throughout this paper. Define $\bar{G}_1$ and $\bar{G}_j(\cdot|\underline{y}_{j-1})$ similarly.

<u>Standard Construction</u>: Let $I\{H\}$ be the inverse function of a survival function H [that is, $I\{H\}(u) \equiv \inf\{t \, \varepsilon \, R: \ H(t) > u\}$, $u \, \varepsilon \, (0,1)]$ and $\underline{U} = (U_1, U_2, \ldots)$ be an n-vector with independent components uniformly distributed in (0,1). Construct $\hat{\underline{X}} = (\hat{X}_1, \hat{X}_2, \ldots)$ such that

$$\hat{X}_1 = I\{\bar{F}_1\}(U_1)$$

and given $\hat{X}_1 = x_1, \ \hat{X}_2 = x_2, \ldots, \hat{X}_{j-1} = x_{j-1}$,

$$\hat{X}_j = I\{\bar{F}_j(\cdot | \underline{x}_{j-1})\} (U_j), \ j \geqq 2 \ .$$

Then one has [e.g. Arjas and Lehtonen (1978)].

Lemma 1.2:   Let $\hat{\underline{X}}$ be the values obtained through the standard construction.

Then $\underline{X} \stackrel{st}{=} \hat{\underline{X}}$.

Using the standard construction, Lemma 1.1 and 1.2, Arjas and Lehtonen (1978) [also see Veinott (1965)] have obtained sufficient condition on $\bar{F}$ and $\bar{G}$ that satisfy (1.2). Specifically they have

Lemma 1.3:   Suppose

$$(1.3) \quad \begin{cases} \bar{F}_1(t) \geqq \bar{G}_1(t) \ , \quad t \ \varepsilon \ R \\ \text{and} \\ \bar{F}_j(t | \underline{x}_{j-1}) \geqq \bar{G}_j(t | \underline{y}_{j-1}) \ , \quad t \ \varepsilon \ R \ , \\ \qquad\qquad x_i > y_i, \ i=1,2,\ldots,j-1; \ j \geq 2 \ . \end{cases}$$

Then

$$(1.4) \qquad\qquad \underline{X} \geqq_{st} \underline{Y} \ .$$

One may easily verify that if $\hat{\underline{X}}$ and $\hat{\underline{Y}}$ are constructed using a common $\underline{U}$ for both in the standard construction, $\hat{\underline{X}} \geqq \hat{\underline{Y}}$. Conditions different from (1.3) that imply (1.4) have been obtained using the NHPP construction [see Shaked and Shanthikumar (1984)] and total hazard construction [see Norros

(1984), Shaked and Shanthikumar (1985)]. In this paper we provide conditions weaker than (1.3) that imply (1.4) and prove it using the standard construction [see Section 2]. Using these results we obtain extensions of (i) a result of Block, Beuno, Savits and Shaked (1984) concening the stochastic monotonicity of independent and identically distributed [i.i.d] random variables conditioned on their partial order statistics, and (ii) a theorem of Efron (1965) regarding an increasing property of Polya frequency functions in Sections 3 and 4, respectively. Finally in Section 5 we establish the negative association of i.i.d random variables conditioned on the partial order statistics and point out a sample application for results in Section 3.

The following preliminaries will be required in Sections 3,4, and 5.

Definition 1.1: A random variable Z or its density function h is said to be Polya frequency function of order two [or log-concave and written PF_2] if $h(s+x)/h(u+x)$ is decreasing in $x \in R$ for all $s > u$ [Karlin (1965)].

The following closure property of PF_2 densities will be needed later.

Lemma 1.5: A convolution of two PF_2 densities is PF_2.

Definition 1.2: Two random variables X and Y with density functions f and g are said to be ordered in the sense of likelihood ratio [and written $X \geq_{\ell r} Y$] if $f(t)/g(t)$ is increasing in $t \in R$ [Karlin (1965)].

Lemma 1.6: Suppose $X \geq_{\ell r} Y$. Then $X \geq_{st} Y$.

Definition 1.3: A random vector $\underline{T} = (T_1, T_2, \ldots, T_n)$ is said to be negatively dependent through stochastic ordering [NDS] if $\{(T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n)| T_i = t\}$ stochastically decreases in t for all values of $i = 1, 2, \ldots, n$.

Block, Savits and Shaked (1985) then show

<u>Lemma 1.7</u>: If $\underline{T}$ is NDS, then

$$P\{T_1 > t_1, \ldots, T_n > t_n\} \leq \prod_{i=1}^{n} P\{T_i > t_i\}$$

and

$$P\{T_1 \leq t_1, \ldots, T_n \leq t_n\} \leq \prod_{i=1}^{n} P\{T_i \leq t_i\} \quad .$$

2. <u>CONDITIONS FOR STOCHASTIC ORDERING OF RANDOM VECTORS</u>

In this section we provide conditons weaker than (1.3) that imply (1.4) and prove it using the standard construction. For a given pair of survival functions $\bar{F}$ and $\bar{G}$ define

$$\alpha_1(t) = I\{\bar{F}_1\}(\bar{G}_1(t)), \quad t \in R \text{ and}$$

$$\alpha_j(\underline{y}_{j-1}, t) = I\{\bar{F}_j(\cdot \mid \underline{\alpha}_{j-1}(\underline{y}_{j-1}))\}(\bar{G}_j(t \mid \underline{y}_{j-1})),$$

$$y_1, y_2, \ldots, y_{j-1}, t \in R \ ; \ j \geq 2 \ ,$$

where $\underline{\alpha}_{j-1}(\underline{y}_{j-1})$ and $\underline{y}_{j-1}$ will have different number of components depending on where it is used. $\underline{\alpha}_{j-1}(\underline{y}_{j-1})$ in $\bar{F}_j(\cdot \mid \underline{\alpha}_{j-1}(\underline{y}_{j-1}))$ is $(\alpha_1(\underline{y}_1), \alpha_2(\underline{y}_2), \ldots, \alpha_{j-1}$ $(\underline{y}_{j-1}))$, $\underline{y}_i$ in $\alpha_i(\underline{y}_i)$ is $(y_1, y_2, \ldots, y_i)$ and $\underline{y}_{j-1}$ in $\alpha_j(\underline{y}_{j-1}, t)$ is $(y_1, y_2, \ldots, y_{j-1})$. Then

<u>Theorem 2.1</u>: Suppose

$$(2.1) \quad \begin{cases} \bar{F}_1(t) \geq \bar{G}_1(t) \quad , \quad t \in R \\ \text{and} \\ \bar{F}_j(t \mid \underline{\alpha}_{j-1}(\underline{y}_{j-1})) \geq \bar{G}_j(t \mid \underline{y}_{j-1}), \ y_1, \ldots, y_{j-1}, t \in R; \end{cases}$$

Then $\underline{X} \geq_{st} \underline{Y}$.

Proof: Constructing two random vectors $\hat{\underline{X}}$ and $\hat{\underline{Y}}$ according to the standard construction with a common $\underline{U}$ one sees that

$$(2.2) \quad \hat{X}_j = \alpha_j(\hat{Y}_1, \hat{Y}_2, \ldots, \hat{Y}_j), \ j \geq 1.$$

With Condition (2.1) one can easily verify that

$$(2.3) \quad \begin{cases} \alpha_1(t) \geq t, \ t \ \varepsilon \ R \\ \alpha_j(\underline{y}_{j-1}, t) \geq t \ , \ y_1, \ y_2, \ldots, y_{j-1}, \ t\varepsilon R; \ j \geq 2 \ . \end{cases}$$

From (2.2) and (2.3) one sees that $\hat{\underline{X}} \geq \hat{\underline{Y}}$ and the result now follows from Lemmas 1.1 and 1.2.

Remark 2.1: From (2.3) it is immediately clear that (1.3) implies (2.1). The popularity of using Lemma 1.3 for stochastic comparison of random vectors is its relative easiness to verify (1.3). As we will see in Section 3 there are interesting examples which satisfy (2.1) but do not satisfy (1.3). In such an example we will also see that it is not hard to verify (2.1).

In some applications it is possible to identify stronger conditions on $\bar{F}$ and $\bar{G}$ that imply (2.1). Such a condition is given in [see Remarks 2.2 and 2.3]:

Theorem 2.2:   Suppose there exist a set of non-negative functions $\{b_1, b_2, \ldots\}$ such that

$$(2.4) \quad \begin{cases} \quad \bar{F}_1(t) \geq \bar{G}_1(t) \geq \bar{F}_1(t+b_1), \ t \ \varepsilon R \\ \\ \text{and} \quad \bar{F}_j(t|\underline{x}_{j-1}) \geq \bar{G}_j(t|\underline{y}_{j-1}) \geq \bar{F}_j(t + b_j(\underline{x}_{j-1}, \underline{y}_{j-1})|\underline{x}_{j-1}), \\ \\ \text{for} \quad t \ \varepsilon \ R, \ y_1 + b_1 \geq x_1 \geq y_1, \ y_i + b_i(\underline{x}_{i-1}, \underline{y}_{i-1}) \geq \\ \\ \qquad x_i \geq y_i, \ i = 1,2,\ldots,j-1; \ j \geq 2, \end{cases}$$

where b_1 is a constant and $b_i (\underline{x}_{i-1}, \underline{y}_{i-1}) \equiv b_i(x_1, x_2, \ldots, x_{i-1}, y_1, y_2, \ldots, y_{i-1})$, $i = 2,3,\ldots$ Then $\underline{X} \geq_{st} \underline{Y}$.

Proof: Observe that $\bar{F}_1(t) \geqq \bar{G}_1(t) \geqq \bar{F}_1(t+b_1)$, $t \, \varepsilon \, R$ implies $t + b_1 \geqq$

$\alpha_1(t) \geqq t$, $t \, \varepsilon \, R$. Therefore $\bar{F}_2(t|x_1) \geqq \bar{G}_2(t|y_1) \geqq \bar{G}_2(t + b_2(x_1,y_1)|x_1)$,

$t \, \varepsilon \, R$, $y_1 + b_1 \geqq x_1 \geqq y_1$ implies

(2.5) $\bar{F}_2(t|\alpha_1(y_1)) \geqq \bar{G}_2(t|y_1) \geqq \bar{F}_2(t + b_2(\alpha_1(y_1),y_1)|\alpha_1(y_1))$, y_1, $t \varepsilon R$.

and hence

(2.6) $t + b_2(\alpha_1(y_1),y_1) \geqq \alpha_2(y_1,t) \geqq t$, y_1, $t \, \varepsilon \, R$.

Now as an induction hypothesis assume that

(2.7) $\bar{F}_i(t|\underline{\alpha}_{i-1}(\underline{y}_{i-1})) \geqq \bar{G}_i(t|\underline{y}_{i-1}) \geqq \bar{F}_i(t+b_i(\underline{\alpha}_{i-1}(\underline{y}_{i-1}),\underline{y}_{i-1})|\underline{\alpha}_{i-1}(\underline{y}_{i-1}))$

and hence

(2.8) $t + b_i(\underline{\alpha}_{i-1}(\underline{y}_{i-1}),\underline{y}_{i-1}) \geqq \alpha_i(\underline{y}_{i-1},t) \geqq t$, y_1, $y_2,\ldots,y_{i-1}$, $t \, \varepsilon \, R$,

for all $i = 2,3,\ldots,j-1$. Note that (2.7) and (2.8) are valid for $i = 2$

[see (2.5) and (2.6)]. Now from (2.8) with $i = j-1$ and (2.4) one

sees that (2.7) is satisfied with $i = j$. However, (2.7) with $i = j$

implies (2.8) with $i = j$. Then by the induction hypothesis one sees

that (2.7) and (2.8) are true for all values of $i = 2,3,\ldots,n$. That is

Condition (2.1) of Theorem 2.1 is satisfied. The result now follows by

Theorem 2.1.

Remark 2.2: As we have noted in the proof of Theorem 2.2, Condition

(2.4) implies (2.1).

Remark 2.3: If we set $b_j = \infty$, $j \geqq 1$, Condition (2.4) reduces to Condition

(1.3). We will use the above result to obtain an extension of a Theorem of

Efron (1965).

3. STOCHASTIC MONOTONICITY OF i.i.d RANDOM VARIABLES

CONDITIONED ON ORDER STATISTICS

Let $\underline{Z}_j = (Z_1, Z_2, \ldots, Z_j)$ be a random vector of j i.i.d random variables with survival function $\bar{H}$ and density h. Now let

$$N_j(t) = \#\{i: Z_i > t, \ i = 1, 2, \ldots, j\}.$$

$\{N_j(t)\}$ is assumed to be right continuous with left hand limits. For a given $r \geq 1$, $\underline{z} = (z_1, z_2, \ldots, z_r)$, $\underline{\ell} = (\ell_1, \ell_2, \ldots, \ell_r)$ and $\underline{m} = (m_1, m_2, \ldots, m_r)$ such that $z_1 < z_2 < \ldots < z_r$ and $0 \leq \ell_r \leq m_r \leq \ldots \leq \ell_2 \leq m_2 \leq \ell_1 \leq m_1 \leq j$, define $\underline{Z}_{j,\underline{z},\underline{\ell},\underline{m}} = \{\underline{Z}_j \mid N_j(z_i) = \ell_i, \ N_j(z_i-) = m_i, \ i = 1, 2, \ldots, r\}$

Let $\underline{T} \equiv \underline{Z}_{j,\underline{z},\underline{\ell},\underline{m}}$. Then simple calculation shows that

$$(3.1) \qquad P\{T_1 > t\} = \frac{1}{j}\{m_i + (\ell_{i-1} - m_i)\bar{H}_{\underline{z},i}(t)\} \,,$$

where

$$\bar{H}_{\underline{z},i}(t) = \frac{\bar{H}(t) - \bar{H}(z_i)}{\bar{H}(z_{i-1}) - \bar{H}(z_i)} \,, \qquad z_{i-1} < t < z_i,$$

$$i = 1, 2, \ldots, r + 1$$

and $z_0 = -\infty$, $z_{r+1} = +\infty$. Since $\bar{H}_{\underline{z},i}(t)$ increases as z_{i-1} and z_i increases for $z_{i-1} < t < z_i$, it is not hard to see

Lemma 3.1: $P\{T_1 > t\}$ given by (3.1) is increasing in $(\underline{z}, \underline{\ell}, \underline{m})$.

Now consider $P\{T_2 > t \mid T_1 = t_1\}$. A routine calculation shows that

$$(3.2) \qquad P\{T_2 > t \mid T_1 = t_1\} = \frac{1}{j-1}\{m_i' + (\ell_{i-1}' - m_i')\bar{H}_{\underline{z},i}(t)\},$$

$$z_{i-1} < t < z_i, \ i = 1, 2, \ldots, r + 1,$$

where

$$(3.3) \quad \begin{cases} \ell'_i = \ell_i - 1_{(z_i < t_1)} \\ \text{and} \\ m'_i = m_i - 1_{(z_i \leqq t_1)} \end{cases} , \quad i = 1,2,\ldots,r.$$

Define $\underline{T}' = (T'_1, T'_2, \ldots T'_{j-1}) = \underline{Z}_{j-1,\underline{z},\underline{\ell}',\underline{m}'}$. Then comparing the right hand side of (3.2) and that of (3.1) one sees that $P\{T_2 > t | T_1 = t_1\} = P\{T'_1 > t\}$, $t \varepsilon R$. Now considerng $P\{T_3 > t | T'_1 = t_2\}$ and continuing this one can establish

__Lemma 3.2:__ $\{(T_2, T_3, \ldots, T_j) | T_1 = t_1\} \overset{st}{=} Z_{j-1,\underline{z},\underline{\ell}',\underline{m}'}$, where $\underline{\ell}'$ and m' are given in (3.3).

__Remark 3.1:__ Note that the event $\{N_j(z_i) = \ell_i, N_j(z_i-) = m_i, i = 1,2,\ldots,r\}$ relays information on the order statistics $z_{[1]} \leqq Z_{[2]} \leqq \ldots \leqq Z_{[j]}$ of $\underline{Z}_j$. For example consider the following event $[1 \leqq k_1 < k_2 < \ldots < k_r \leqq j]$:

$$(3.4) \quad A = \{N_j(z_i) = j - k_i, N_j(z_i-) = j - k_i + 1, i = 1,2,\ldots,r\},$$
$$\text{for some } 1 \leqq r \leqq j.$$

Then clearly

$$A \equiv \{Z_{[k_1]} = z_1, Z_{[k_2]} = z_2, \ldots, Z_{[k_r]} = z_r\}.$$

Block, Bueno, Savits and Shaked (1984) consider the random vector $\underline{Z}_j$ conditioned on event A as given in (3.4) and study its stochastic monotonicity and NDS properties. Alternatively consider

$$(3.5) \quad A = \{N_j(z_i) = j - k_i + 1 = N_j(z_i-), i = 1,2,\ldots,r\}.$$

Then

$$A \equiv \{Z_{[k_i-1]} < z_i < Z_{[k_i]}, i = 1,2,\ldots,r\}, \text{ where we set } Z_{[0]} = -\infty.$$

Now we present the main theorem of this section.

<u>Theorem 3.3:</u> Let $\underline{Z}_{n,\underline{z},\underline{\ell},\underline{m}}$ be as defined in this section with $j = n$. Then $\underline{Z}_{n,\underline{z},\underline{\ell},\underline{m}}$ is stochastically increasing in $(\underline{z}, \underline{\ell}, \underline{m})$ [that is $(\underline{w},\underline{u},\underline{v}) \geqq (\underline{c},\underline{a},\underline{b}) \Rightarrow \underline{Z}_{n,\underline{w},\underline{u},\underline{v}} \geqq_{st} \underline{Z}_{n,\underline{c},\underline{a},\underline{b}}$].

<u>Proof:</u> Let $\bar{F}$ and $\bar{G}$ be the survival functions of $\underline{X} \equiv \underline{Z}_{n,\underline{w},\underline{u},\underline{v}}$ and $\underline{Y} \equiv \underline{Z}_{n,\underline{c},\underline{a},\underline{b}}$, respectively for some $(\underline{w},\underline{u},\underline{v}) \geqq (\underline{c},\underline{a},\underline{b})$. From Lemma 3.1 one has

$$(3.6) \qquad \bar{F}_1(t) \geqq \bar{G}_1(t), \quad t \, \varepsilon \, R$$

Now consider $\alpha_1(t)$ as defined in Section 2. From (3.6) it is clear that $\alpha_1(t) \geqq t$. A careful study of $\bar{F}_1$ and $\bar{G}_1$ [given in (3.1) with appropriate values for $\underline{z}$, $\underline{\ell}$ and $\underline{m}$] will show that for fixed i, c_i and w_i:

$$(3.7) \left\{ \begin{array}{l} \text{If } u_i = a_i \text{ then } \{t : t > c_i\} = \{t : \alpha_1(t) > w_i\}, \\[1em] \text{If } u_i > a_i \text{ then } \{t : t > c_i\} \subset \{t : \alpha_1(t) > w_i\}, \\[2em] \text{if } v_i = b_i \text{ then } \{t : t \geqq c_i\} \equiv \{t : \alpha_1(t) \geqq w_i\}, \text{ and} \\[1em] \text{if } v_i > b_i \text{ then } \{t : t \geqq c_i\} \subset \{t : \alpha_1(t) \geqq w_i\}. \end{array} \right.$$

Therefore if we set

$$u_i' = u_i - 1_{(w_i < \alpha_1(y_1))} \; ; \; v_i' = v_i - 1_{(w_i \leqq \alpha_1(y_1))}$$

and

$$a_i' = a_i - 1_{(c_i < y_i)} \; ; \; b_i' = b_i - 1_{(c_i \leqq y_i)} ,$$

from (3.7) one sees that for all $y_1 \, \varepsilon \, R$, $(\underline{u}', \underline{v}') \geqq (\underline{a}', \underline{b}')$. Therefore from Lemmas 3.1 and 3.2 one has

$$\{X_2, X_3, \ldots, X_n | X_1 = \alpha_1(y_1)\} \stackrel{st}{=} \underline{Z}_{n-1,\underline{w},\underline{u}',\underline{v}'} \quad ,$$

$$\{Y_2, Y_3, \ldots, Y_n | Y_1 = y_1\} \stackrel{st}{=} \underline{Z}_{n-1,\underline{c},\underline{a}',\underline{b}'} \quad ,$$

and

$$\bar{F}_2(t | \alpha_1(y_1)) \geqq \bar{G}_2(t | y_1) \ , \ y_1, \ t \ \varepsilon \ R.$$

Therefore a continued application of the above analysis will result in

$$\bar{F}_j(t | \underline{\alpha}_{j-1}(\underline{y}_{j-1})) \geqq \bar{G}_j(t | \underline{y}_{j-1}), \ y_1, y_2, \ldots, y_{j-1}, \ t \ \varepsilon \ R \ ; \quad j \geqq 2 \quad .$$

That is $(\bar{F}_j)$ and $(\bar{G}_j)$ satisfy Condition (2.1) of Theorem 2.1 and there-fore $\underline{X} \geqq_{st} \underline{Y}$.

Now combining (3.4) with the above result one obtains

<u>Corollary 3.4</u>: For some $1 \leqq r \leqq n$, $z_1 < z_2 < \ldots < z_r$, and $1 \leqq k_1 < k_2 < \ldots$ $< k_r \leqq n$ let $\underline{T} \equiv \{\underline{Z}_n | Z_{[k_1]} = z_1, \ Z_{[k_2]} = z_2, \ldots, Z_{[k_r]} = z_r\}$. Then $\underline{T}$ is (i) stochastically increasing in $\underline{z}$ and (ii) stochastically decreasing in $\underline{k}$.

Block, Beuno, Savits and Shaked (1984) proved (i) [see the Corollary to Lemma 4.2 there]. They also used this result to show some negative dependence properties of $\underline{T}$. In Section 5 we will show the negative dependence property of $\underline{Z}_{n,\underline{z},\underline{\ell},\underline{m}}$. Now combining (3.5) with Theorem 3.3 one obtains [as a special case].

<u>Corollary 3.5</u>: For some $1 \leq k \leq n$ and $z \in R$ let $\underline{T} \equiv \{\underline{Z}_n | Z_{[k-1]} < z <$ $Z_{[k]}\}$. Then $\underline{T}$ is stochastically increasing in z and is decreasing in k.

We will use the above result to provide a decision rule for selecting components during cannibalization [see Section 5].

4. INCREASING PROPERTY OF PÓLYA FREQUENCY FUNCTIONS

Let $\underline{Z} = (Z_1, Z_2, \ldots, Z_n)$ be n mutually independent PF_2 random variables with density functions $(h_1, h_2, \ldots, h_n)$. Let $\underline{Z}^s \equiv \{\underline{Z} | \underline{Z} \underline{1} = s\}$, where $\underline{1}' = (1, 1, \ldots, 1)$ is of appropriate dimension.

<u>Theorem 4.1</u>: For some $s \geq u$, let $\bar{F}$ and $\bar{G}$ be the survival functions of $\underline{Z}^s$ and $\underline{Z}^u$ respectively. Then (i) $\bar{F}$ and $\bar{G}$ satisfy Condition (2.4) of Theorem 2.2, and (ii) $\underline{Z}^s \geq_{st} \underline{Z}^u$.

<u>Proof</u>: From Theorem 2.2 and (i), (ii) follows. So consider (i). Let q_j be the density function of the sum $Z_j + Z_{j+1} + \ldots + Z_n$. Then if (f_j) and (g_j) are the densities of $(\bar{F}_j)$ and $(\bar{G}_j)$ respectively,

$$\frac{f_1(t)}{g_1(t)} = \left\{ \frac{q_2(s-t)}{q_2(u-t)} \right\} \left\{ \frac{q_1(u)}{q_1(s)} \right\}$$

Note that $s-t \geq u-t$ and since q_j is PF_2 [Lemma 1.5], $q_2(s-t)/q_2(u-t)$ and hence $f_1(t)/g_1(t)$ increases in t. That is $X_1 \geq_{\ell r} Y_1$ and from Lemma 1.6, one has

$$\bar{F}_1(t) \geq \bar{G}_1(t) \quad , \quad t \in R \quad . \text{ Now consider}$$

$$\frac{f_1(t + (s-u))}{g_1(t)} = \left\{ \frac{h_1(t + (s-u))}{h_1(t)} \right\} \left\{ \frac{q_1(u)}{q_1(s)} \right\}$$

Since h_1 is PF_2, $h_1(t-(s-u))/h_1(t)$ and hence $f_1(t-(s-u))/g_1(t)$ decreases in t. Therefore as before $Y_1 \geq_{\ell r} X_1-(s-u)$ and

$$(4.1) \qquad \bar{F}_1(t) \geq \bar{G}_1(t) \geq \bar{F}_1(t + b_1) \quad,$$

where

$$b_1 \equiv s-u \ .$$

Now let

$$b_j(\underline{x}_{j-1}, \underline{y}_{j-1}) \equiv s-u- \sum_{i=1}^{j-1} (x_i - y_i) \quad, \ j = 2,3,\ldots,n.$$

From (4.1) one has $y_1 + b_1 \geq \alpha_1(y_1) \geq y_1$. Hence with restriction $y_1 + b_1 \geq x_1 \geq y_1$ one sees that

$$(4.2) \qquad b_2(x_1,y_1) \geq 0 \ .$$

Now as an induction hypothesis assume that

(A1) $b_j(\underline{x}_{j-1}, \underline{y}_{j-1}) \geq 0$, $j \leq \ell$ and

(A2) Condition (2.4) holds for all values of j up to ℓ-1.

Note that (A1) and (A2) is true for $\ell = 2$ [see (4.1) and (4.2)]. Consider

$$\frac{f_\ell(t|\underline{x}_{\ell-1})}{g_\ell(t|\underline{y}_{\ell-1}))} = \left\{ \frac{q_{\ell+1}(s-t- \sum_{i=1}^{\ell-1} x_i)}{q_{\ell+1}(u-t- \sum_{i=1}^{\ell-1} y_i)} \right\} \left\{ \frac{q_\ell(u- \sum_{i=1}^{\ell-1} y_i)}{q_\ell(s- \sum_{i=1}^{\ell-1} x_i)} \right\}$$

Since by Assumption (A1), $b_\ell(\underline{x}_{\ell-1}, \underline{y}_{\ell-1}) \geq 0$, one has

$$s-t- \sum_{i=1}^{\ell-1} x_i \geq u - t - \sum_{i=1}^{\ell-1} y_i \quad .$$

Therefore as before

$$\{X_\ell | X_1 = x_1, \ldots, X_{\ell-1} = x_{\ell-1}\} \geq_{\ell r} \{Y_\ell | Y_1 = y_1, \ldots, Y_{\ell-1} = y_{\ell-1}\} \, ,$$

$$y_i + b_i(\underline{x}_{i-1}, \underline{y}_{i-1}) \geq x_i \geq y_i, \quad i = 1, 2, \ldots, \ell-1, \text{ and}$$

$$\bar{F}_\ell(t | \underline{x}_{\ell-1}) \geq \bar{G}_\ell(t | \underline{y}_{\ell-1}) \, .$$

Now consider

$$\frac{f_\ell(t + b_\ell(\underline{x}_{\ell-1}, \underline{y}_{\ell-1}) | \underline{x}_{\ell-1})}{g_\ell(t | \underline{y}_{\ell-1})} = \left\{ \frac{h_\ell(t + b_\ell(\underline{x}_{\ell-1}, \underline{y}_{\ell-1}))}{h_\ell(t)} \right\} \left\{ \frac{q_\ell(u - \sum_{i=1}^{\ell-1} y_i)}{q_\ell(s - \sum_{i=1}^{\ell-1} x_i)} \right\}$$

Since by Assumption (A1) $b_\ell(\underline{x}_{\ell-1}, \underline{y}_{\ell-1}) \geq 0$, the log concavity of h_ℓ implies

(4.3) $\bar{F}_\ell(t | \underline{x}_{\ell-1}) \geq \bar{G}_\ell(t | \underline{y}_{\ell-1}) \geq \bar{F}_\ell(t + b_\ell(\underline{x}_{\ell-1}, \underline{y}_{\ell-1}) | \underline{x}_{\ell-1}) \, ,$

for all $y_1 + b_1 \geq x_1 \geq y_1$, $y_i + b_i(\underline{x}_{i-1}, \underline{y}_{i-1}) \geq x_i \geq y_i$, $i = 1, 2, \ldots, \ell-1$. With (4.3) one has

(4.4) $y_\ell + b_\ell(\underline{x}_{\ell-1}, \underline{y}_{\ell-1}) \geq \alpha_\ell(\underline{y}_\ell) \geq y_\ell \, .$

Therefore with the restriction $y_1 + b_1 \geq x_1 \geq y_1$, $y_i + b_i(\underline{x}_{i-1}, \underline{y}_{i-1}) \geq x_i \geq y_i$, $i = 1, 2, \ldots, \ell-1$ one obtains

(4.5) $b_{\ell+1}(\underline{x}_\ell, \underline{y}_\ell) = b_\ell(\underline{x}_{\ell-1}, \underline{y}_{\ell-1}) - (x_\ell - y_\ell) \geq 0.$

From (4.3) and (4.5) and the induction hypothesis one sees that $\bar{F}$ and $\bar{G}$ satisfy Condition (2.4).

Remark 4.1: Efron (1965) using an alternative proof established (ii) of Theorem 4.1. The discrete analogue of this result has proved to be very useful in the analysis of queueing networks [see Shanthikumar and Yao (1985a), (1985b)]. We will next provide an extension [the main result of this section] of Theorem 4.1. This extension has useful applications in the analysis of queueing network and these applications will be discussed elsewhere.

Theorem 4.2: Let $\hat{\underline{X}} = (\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n)$ and $\hat{\underline{Y}} = (\hat{Y}_1, \hat{Y}_2, \ldots, \hat{Y}_n)$

be two independent random vectors of mutually independent elements

with PF_2 densities $(\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n)$ and $(\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_n)$ respectively.

For some s and u define

$$\underline{X} = \{\hat{\underline{X}} | \hat{\underline{X}}\,1 = s\} \text{ and } \underline{Y} = \{\hat{\underline{Y}} | \hat{\underline{Y}}\,1 = u\} .$$

Suppose for some $0 \leq a_i < \infty$

$$\hat{X}_i \geq_{\ell r} \hat{Y}_i \geq_{\ell r} \hat{X}_i - a_i , \quad i = 1, 2, \ldots, n.$$

Then for $s \geq u + \sum_{i=1}^{n} a_i$, $\underline{X} \geq_{st} \underline{Y}$.

Proof: Suppose we can show that

$$(4.6) \qquad \{\hat{X}_1, \hat{Y}_2, \ldots, \hat{Y}_n | \hat{X}_1 + \hat{Y}_2 + \ldots + \hat{Y}_n = v\} \geq_{st}$$

$$\{\hat{Y}_1, \hat{Y}_2, \ldots, \hat{Y}_n | \hat{Y}_1 + \hat{Y}_2 + \ldots + \hat{Y}_n = u\} , \quad v \geq u + a_1 .$$

Then applying (4.6) with $\hat{X}_2$ replacing $\hat{Y}_2$ one sees that

$$\{\hat{X}_1, \hat{X}_2, \hat{Y}_3, \ldots, \hat{Y}_n | \hat{X}_1 + \hat{X}_2 + \hat{Y}_3 + \ldots + \hat{Y}_n = w\}$$

$$\geq_{st} \{\hat{X}_1, \hat{Y}_2, \hat{Y}_3, \ldots, \hat{Y}_n | \hat{X}_1 + \hat{Y}_2 + \hat{Y}_3 + \ldots + \hat{Y}_n = v\}, \quad w \geq v + a_2$$

$$\geq_{st} \{\hat{Y}_1, \hat{Y}_2, \hat{Y}_3, \ldots, \hat{Y}_n | \hat{Y}_1 + \hat{Y}_2 + \hat{Y}_3 + \ldots + \hat{Y}_n = u\},$$

$$w \geq v + a_2$$

$$\geq u + a_1 + a_2.$$

Continuing this way one can obtain the desired result. Hence all we need

is to establish (4.6) for $v = u + a_1$. So without loss of generality let the

left hand side of (4.6) be $\underline{X}$ and the right hand side be $\underline{Y}$, with survival

functions $\bar{F}$ and $\bar{G}$, and densities f and g, respectively. Then,

$$\frac{f_1(t)}{g_1(t)} = \left\{\frac{\hat{f}_1(t)}{\hat{g}_1(t)}\right\} \left\{\frac{q_2(v-t)}{q_2(u-t)}\right\} \left\{\frac{\hat{g}_1 * q_2(u)}{\hat{f}_1 * q_2(v)}\right\}$$

where q_2 is the PF_2 density of $\sum\limits_{i=2}^{n} \hat{Y}_i$ and $*$ stands for convolution.

As before one sees that $f_1(t)/g_1(t)$ is increasing in t and hence

$$\bar{F}_1(t) \geqq \bar{G}_1(t) \ , \ t \ \varepsilon \ R \ .$$

Consider

$$\frac{f_1(t + a_1)}{g_1(t)} = \left\{\frac{\hat{f}_1(t + a_1)}{\hat{g}_1(t)}\right\} \left\{\frac{\hat{g}_1 * q_2(u)}{\hat{f}_1 * q_2(v)}\right\}$$

The above expression is decreasing in t and hence

$$\bar{F}_1(t) \geqq \bar{G}_1(t) \geqq \bar{F}_1(t + a_1).$$

So $\bar{F}_1$ and $\bar{G}_1$ satisfy (2.4) with $b_1 =. a_1 \geqq 0$. Now note that

$$\{X_1, X_2, \ldots, X_n | X_1 = x_1\} =$$
$$\{\hat{Y}_2, \hat{Y}_3, \ldots, \hat{Y}_n | \hat{Y}_2 + \hat{Y}_3 + \ldots + \hat{Y}_n =$$
$$v - x_1\} \equiv \underline{Y}^{v-x}{}_1$$

and similarly

$$\{Y_2, Y_3, \ldots, Y_n | Y_1 = y_1\} =$$
$$\{\hat{Y}_2, \hat{Y}_3, \ldots, \hat{Y}_n | \hat{Y}_2 + \hat{Y}_3 + \ldots + \hat{Y}_n =$$
$$u - y_1\} \equiv \underline{Y}^{u-y}{}_1 \ .$$

Then for $y_1 + b_1 \geqq x_1 \geqq y_1$, $v - x_1 \geqq u - y_1$, and therefore from Theorem 4.1 one sees that $\bar{F}_j$ and $\bar{G}_j$, $j \geqq 2$ also satisfy Condition (2.4) of Theorem 2.2. Hence $\underline{X} \geqq_{st} \underline{Y}$.

<u>Remark 4.2</u>: Obviously when $\hat{X}_i \overset{st}{=} \hat{Y}_i$, one can set $a_i = 0$, $i = 1, 2, \ldots, n$. Then Theorem 4.2 specializes to Theorem 4.1. Now suppose instead of conditioning $\hat{\underline{X}} 1 = s$ and $\underline{\hat{Y}} 1 = u$, we condition them to be $\underline{\hat{X}} 1 \overset{st}{=} Z_1$ and $\underline{\hat{Y}} 1 \overset{st}{=} Z_2$, where Z_1 and Z_2 are two random variables with distri-

butions H_1 and H_2, respectively. That is,

$$P\{\underline{X} \, \varepsilon \, A\} \equiv \int P\{\hat{\underline{X}} \, \varepsilon \, A | \hat{\underline{Y}} \, \underline{1} = s\} \, dH_1(s) \text{ and}$$

$$P\{\underline{Y} \, \varepsilon \, A\} \equiv \int P\{\hat{\underline{Y}} \, \varepsilon \, A \mid \hat{\underline{Y}} \, \underline{1} = u\} \, dH_2(u) \quad , A \, \varepsilon \, R^n \quad .$$

Suppose $Z_1 \geq_{st} Z_2 + \sum\limits_{i=1}^{n} a_i$. Then defining $Z_1' \equiv Z_1 - \sum\limits_{i=1}^{n} a_i$ with

distribution function H_1' one obtains

$$P\{\underline{X} \, \varepsilon \, A\} = \int \mathcal{H} \, (A, \, u) \, dH_1'(u) \text{ and}$$

$$P\{\underline{Y} \, \varepsilon \, A\} = \int\limits_{u} \mathcal{Y}(A,u) dH_2(u) \quad , A \, \varepsilon \, R^n \quad ,$$

where $\quad \mathcal{H} \, (A,u) = P\{\hat{\underline{X}} \, \varepsilon \, A | \hat{\underline{X}} \, \underline{1} = u + \sum\limits_{i=1}^{n} a_i\}$

$$\mathcal{Y}(A,u) = P\{\hat{\underline{Y}} \, \varepsilon \, A | \hat{\underline{Y}} \, \underline{1} = u\}.$$

From Theorem 4.2 one has $\mathcal{H}(A,u) \geq \mathcal{Y}(A,u)$ for all increasing upper sets A, and from Theorem 4.1 one has $\mathcal{H}(A,u)$ and $\mathcal{Y}(A,u)$ increasing in u.

Then it is easily seen that since $Z_1 \geq_{st} Z_2 + \sum\limits_{i=1}^{n} a_i$ [that is $H_1'(u) \leq H_2(u)$], one has $P\{\underline{X} \, \varepsilon \, A\} \geq P\{\underline{Y} \, \varepsilon \, A\}$ for all increasing sets $A \, \varepsilon \, R^n$. Therefore $\underline{X} \geq_{st} \underline{Y}$.

Using analysis similar to that in this section it can be shown that all the results presented in this section holds true for discrete random variables. A special case of the above extension has been used by Shanthikumar and Yao (1985b) in the analysis of closed queueing networks.

5. APPLICATIONS

We will first apply Theorem 3.3 to show that $\underline{Z}_{n,\underline{z},\underline{\ell},\underline{m}}$ defined in Section 3 satisfies the NDS property of Block, Savits and Shaked (1985) [see Definition 1.5].

<u>Theorem 5.1:</u> $\underline{T} \equiv \underline{Z}_{n,\underline{z},\underline{\ell},\underline{m}}$ satisfy the NDS propetry.

Proof: From Lemma 3.2 it can be seen that

(5.1) $\{(T_1, T_2, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n) | T_i = s\} \stackrel{st}{=} \underline{Z}_{n-1,\underline{z},\underline{\ell}^s,\underline{m}^s}$, where

$\ell_1^s = \ell_i - I_{(z < s)}$ and $m_i^s = m_i - I_{(z \leq s)}$, $i = 1,2,\ldots,n$. Therefore

$s > t = (\underline{\ell}^s, \underline{m}^s) \leq (\underline{\ell}^t, \underline{m}^t)$. Then from (5.1) and Theorem 3.3 one

has the NDS property for $\underline{T}$.

Next we point out an application for Corollary 3.5 in component cannibali-
zation. Consider a collection of heat sources each cooled by its own cooling
system consisting of a set of n identical pumps and a circulation system
[composed of radiators, pipes etc.]. The operation of the heat source is
continued unless either the heat source or the cooling system fails. A
cooling system failure may occur either because all the n pumps have failed
or because the circulation system has failed. The circulation system fails
mainly due to the structural damage caused by fewer number of pumps
working. One may safely assume that the damage accumulation rate increases
as the number of working pumps decreases.

After t time units of operation all the cooling systems are replaced by
spare cooling systems. This t time units may represent the 'high reliable
operatng time' for the cooling system. These pulled out cooling systems
may however be used elsewhere for less critical use. Since some of the
pumps from these pulled out cooling systems may have already failed one
could pool all such cooling systems and assemble fewer number of systems
but all with n operating pumps. In such a case one has to choose the best
circulation systems for these re-assembly. Since it is not economically
feasible to test the circulation system for its structural damage one may use
the following result to choose the desired circulation systems.

Let C_i be the circulation system belonging to the i-th cooling system.

Assume that we have also observed that k_i of the n pumps of the i-th cooling system were not operative at time t. Then if D_i is the damage accumulated in C_i, one has from Corollary 3.5

(5.1) $\qquad D_{\pi(1)} \leq_{st} D_{\pi(2)} \leq_{st} \cdots \leq D_{\pi(m)}$,

where π is a permutation of $\{1,2,\ldots,m\}$ such that

(5.2) $\qquad k_{\pi(1)} \leq k_{\pi(2)} \leq \cdots \leq k_{\pi(m)}$.

One may now use (5.1) and (5.2) to choose the stochastically better circulation systems for re-assembly.

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